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THE DESCRIPTION OF MATERIAL SYMMETRY IN MATERIALS WITH MEMORY

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Abstract—Two forms of the conventional procedure for describing material symmetry are presented in the context of the mechanics of materials with memory, in which the stress matrix at time t is assumed to be a functional of the history of the deformation gradient matrix in the time interval $[t_0, t]$. In one of these forms the particle considered is identified by its position at time t_0 and in the other by its position at time t. This procedure is contrasted with that proposed by Noll in 1958. Unlike the conventional procedure, in that of Noll the concepts of material symmetry and material frame indifference are inextricably intertwined. Moreover, it does not have general applicability to areas of continuum physics other than mechanics, as does the conventional procedure.

INTRODUCTION

In this paper two ways of giving mathematical expression to material symmetry are compared. One of these has been traditionally used in continuum mechanics and in continuum theories in crystal physics. In the context of continuum–mechanical theories of materials with memory, in which the Cauchy stress at time t is assumed to be a functional of the deformation gradient history in the time interval $[t_0, t]$, it was used by Green and Rivlin[1]. It is formulated in this context in Sections 2 and 3. In Section 2 the particle considered is identified by its position at time t_0, and in Section 3 by its position at time t. In both cases the physical concept of material symmetry is translated directly into a restriction on the constitutive functional. In Sections 4 and 5 it is shown, by two different methods, that if the particle considered is identified by its position at time t and the material considered is isotropic, then the Cauchy stress at time t may be expressed as an isotropic functional of the history of the Cauchy strain referred to the configuration at time t.

Another procedure for giving mathematical expression to material symmetry, again in the context of the continuum mechanics of materials with memory, was introduced by Noll[2] in 1958 and has been reproduced extensively in the secondary literature since that time. In Noll's procedure, which is described in Section 6, an "isotropy group" is defined as the group of transformations for which a particular invariance condition (eqn (57) below) is satisfied by the constitutive functional. Although this is a necessary condition for the group of transformations to describe the material symmetry, it is not a sufficient condition. Sufficiency is obtained only if the constitutive functional is also required to satisfy material frame indifference. Then the restriction on the constitutive functional which is obtained is identical with that obtained directly by the traditional procedure.

We see that in Noll's procedure the concepts of material symmetry and material frame indifference are intertwined, while in the traditional procedure they remain distinct concepts. We illustrate the effect of this in Section 7 by considering the manner in which material symmetry restrictions can be introduced into the constitutive equation for the heat flux vector in a rigid body in which temperature gradients exist. It is seen that while the traditional procedure provides appropriate restrictions on the constitutive equations, Noll's procedure is inapplicable.

In Ref. [2] Noll uses his invariance condition (eqn (57) below) to distinguish between solids and fluids. He supposes that for solids the isotropy group is either the full orthogonal group or a sub-group of it; for fluids it is the full unimodular group. However, it is seen in

‡This paper was written while R.S.R. was a visting professor at the University of Delaware. He wishes to express his gratitude for its hospitality.

Section 6 that Noll's conclusions are incorrect and would, indeed, disallow the spatial description of the deformation discussed in Sections 4 and 5 for an isotropic solid.

2. MATERIAL DESCRIPTION OF MATERIAL SYMMETRY

We consider a body of homogeneous material. We suppose that from it two congruent bodies, B and $\bar{B}$, are cut, as shown schematically in Fig. 1(a). We suppose that the position and spatial orientations of the two bodies are maintained and that x and $\bar{x}$ are two rectangular Cartesian coordinate systems correspondingly located with respect to bodies B and $\bar{B}$, respectively, as shown in Fig. 1(b). We denote the origins of the systems x and $\bar{x}$ by O and $\bar{O}$, respectively, and let $\mathbf{c} = \|c_i\|$ be the column matrix formed by the components of the vector $O\bar{O}$ in the coordinate system x.

We now suppose that the bodies B and $\bar{B}$ are subjected to congruent deformations in the time interval $[t_0, t]$. Let P and $\bar{P}$ be generic particles in bodies B and $\bar{B}$. Let $\mathbf{X}$ $(= \|X_i\|)$ and $\mathbf{x}(\tau)$ $(= \|x_i(\tau)\|)$ be the vector positions of P at times t_0 and τ, respectively, referred to system x. Similarly, let $\bar{\mathbf{X}}$ $(= \|\bar{X}_i\|)$ and $\bar{\mathbf{x}}(\tau)$ $(= \|\bar{X}_i(\tau)\|)$ be the vector positions of $\bar{P}$ at times t_0 and τ, respectively, referred to system $\bar{x}$. Let $\hat{\mathbf{X}}$ $(= \|\hat{\mathbf{X}}_j\|)$ and $\hat{\mathbf{x}}(\tau)$ $(= \|\hat{\mathbf{x}}_i(\tau)\|)$ be the vector positions of $\bar{P}$ referred to coordinate systems x. Then $\hat{\mathbf{X}}$ and $\bar{\mathbf{X}}$, and $\hat{\mathbf{x}}(\tau)$ and $\bar{\mathbf{x}}(\tau)$ are related by transformations of the form

$$\mathbf{X} = \mathbf{S}\bar{\mathbf{X}} + \mathbf{c}, \qquad \hat{\mathbf{x}}(\tau) = \mathbf{S}\bar{\mathbf{x}}(\tau) + \mathbf{c} \tag{1}$$

where S is a constant orthogonal matrix. Matrix S is proper orthogonal if x and $\bar{x}$ are both right-handed or both left-handed; otherwise S is improper orthogonal.

We define the deformation gradient matrix $\mathbf{g}(\tau)$ at P at time τ referred to system x, by

$$\mathbf{g}(\tau) = \|g_{ij}(\tau)\| = \|\partial x_i(\tau)/\partial X_j\|. \tag{2}$$

We denote the deformation gradient matrices at $\bar{P}$ at time τ, referred to the systems $\bar{x}$ and x, by $\bar{\mathbf{g}}(\tau)$ and $\hat{\mathbf{g}}(\tau)$ respectively, and define them by

$$\begin{aligned}\bar{\mathbf{g}}(\tau) &= \|\bar{g}_{ij}(\tau)\| = \|\partial \bar{x}_i(\tau)/\partial \bar{X}_j\| \\ \hat{\mathbf{g}}(\tau) &= \|\hat{g}_{ij}(\tau)\| = \|\partial \hat{x}_i(\tau)/\partial \hat{X}_j\|.\end{aligned} \tag{3}$$

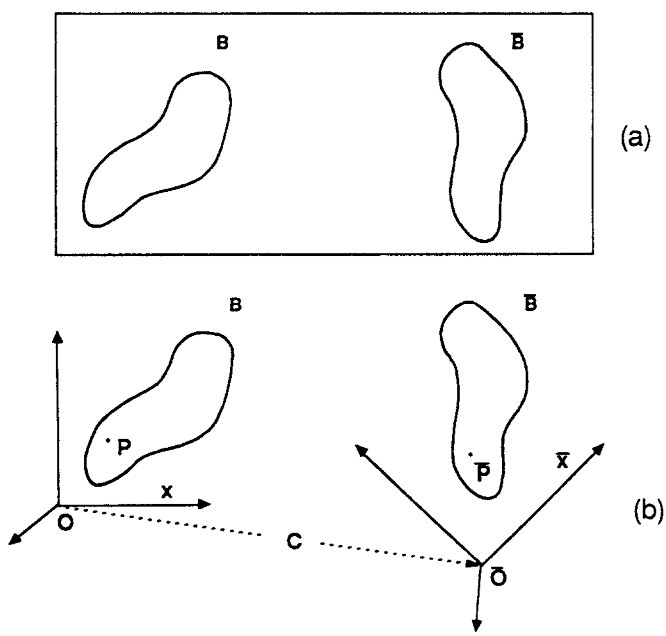

Fig. 1. (a) Congruent domains in a homogeneous block. (b) Congruent bodies cut from a homogeneous block.

Then from eqns (1) we obtain

$$\hat{\mathbf{g}}(\tau) = \mathbf{S}\bar{\mathbf{g}}(\tau)\mathbf{S}^{\dagger} \tag{4}$$

where the dagger denotes the transpose.

We suppose that the deformation of body B is described in system x by

$$\mathbf{x}(\tau) = \mathbf{f}(\mathbf{X}, \tau). \tag{5}$$

Since it has been assumed that the deformations of bodies B and $\bar{B}$ are congruent, it follows that the deformation of body $\bar{B}$ may be described in system $\bar{x}$ by

$$\bar{\mathbf{x}}(\tau) = \mathbf{f}(\bar{\mathbf{X}}, \tau). \tag{6}$$

We make the constitutive assumption that the Cauchy stress at a particle of the material at time τ is determined by the history of the deformation gradient matrix at the particle in the time interval $[t_0, t]$. Let $\boldsymbol{\sigma}(t)$ $(= \|\sigma_{ij}(t)\|)$ and $\bar{\boldsymbol{\sigma}}(t)$ $(= \|\bar{\sigma}_{ij}(t)\|)$ be the Cauchy stress matrices at P and $\bar{P}$ respectively at time t, referred to coordinate systems x and $\bar{x}$ respectively. Then

$$\boldsymbol{\sigma}(t) = \mathbf{F}\{\mathbf{g}(\tau)\}, \qquad \bar{\boldsymbol{\sigma}}(t) = \bar{\mathbf{F}}\{\bar{\mathbf{g}}(\tau)\} \tag{7}$$

and we note that the functionals $\mathbf{F}$ and $\bar{\mathbf{F}}$ are, in general, different. If and only if they are the same, the coordinate systems x and $\bar{x}$ are said to be *equivalent*. Then eqns (7) become

$$\boldsymbol{\sigma}(t) = \mathbf{F}\{\mathbf{g}(\tau)\}, \qquad \bar{\boldsymbol{\sigma}}(t) = \mathbf{F}\{\bar{\mathbf{g}}(\tau)\}. \tag{8}$$

If $\hat{\boldsymbol{\sigma}}(t)$ $(= \|\hat{\sigma}_{ij}(t)\|)$ denotes the Cauchy stress matrix at $\bar{P}$ at time t referred to coordinate system x, then

$$\hat{\boldsymbol{\sigma}}(t) = \mathbf{F}\{\hat{\mathbf{g}}(\tau)\}. \tag{9}$$

From eqns (1) it follows that $\hat{\boldsymbol{\sigma}}(t)$ and $\bar{\boldsymbol{\sigma}}(t)$ are related by

$$\hat{\boldsymbol{\sigma}}(t) = \mathbf{S}\bar{\boldsymbol{\sigma}}(t)\mathbf{S}^{\dagger}. \tag{10}$$

We now suppose that P and $\bar{P}$ are corresponding particles of bodies B and $\bar{B}$; i.e.

$$\bar{\mathbf{X}} = \mathbf{X}, \qquad \bar{\mathbf{x}}(\tau) = \mathbf{x}(\tau). \tag{11}$$

Then, from eqns (2) and (3)

$$\bar{\mathbf{g}}(\tau) = \mathbf{g}(\tau) \tag{12}$$

and from eqns (12) and (4)

$$\hat{\mathbf{g}}(\tau) = \mathbf{S}\mathbf{g}(\tau)\mathbf{S}^{\dagger}. \tag{13}$$

With eqns (12) and (13), it follows from eqns $(7)_2$, (9) and (10) that if P and $\bar{P}$ are corresponding particles of B and $\bar{B}$

$$\mathbf{S}\bar{\mathbf{F}}\{\mathbf{g}(\tau)\}\mathbf{S}^{\dagger} = \mathbf{F}\{\mathbf{S}\mathbf{g}(\tau)\mathbf{S}^{\dagger}\}. \tag{14}$$

If the coordinate systems x and $\bar{x}$ are equivalent, then eqn (14) becomes

$$\mathbf{S}\mathbf{F}\{\mathbf{g}(\tau)\}\mathbf{S}^{\dagger} = \mathbf{F}\{\mathbf{S}\mathbf{g}(\tau)\mathbf{S}^{\dagger}\}. \tag{15}$$

If two rectangular Cartesian coordinate systems are each equivalent to x, they are equivalent to each other. Let $\{x\}$ denote the set of all equivalent coordinate systems which are equivalent to x and let $\{\mathbf{S}\}$ denote the set of all transformations which relate pairs of coordinate systems in this set. Then $\{\mathbf{S}\}$ forms a group of transformations which describes the symmetry of the material. This group is evidently the full orthogonal group or a subgroup of it. Relation (15) must, of course, be satisfied for all, $\mathbf{S}$ in this group.

We now introduce the restriction imposed on $\mathbf{F}$ by the assumption that if an arbitrary time-dependent rigid rotation, which leaves the configuration at time t_0 unchanged, is superposed on the assumed deformation, the Cauchy stress is rotated by the amount of this rotation at time t. This is sometimes called (erroneously) the Principle of Material Frame Indifference. It implies that for body B, functional $\mathbf{F}$ must satisfy the relation

$$\mathbf{F}\{\mathbf{R}(\tau)\mathbf{g}(\tau)\} = \mathbf{R}(t)\mathbf{F}\{\mathbf{g}(\tau)\}\mathbf{R}^{\dagger}(t) \tag{16}$$

for all proper orthogonal matrices $\mathbf{R}(\tau)$ such that $\mathbf{R}(t_0) = \mathbf{I}$. This implicit restriction on the form of $\mathbf{F}$ can be made explicit. We find that

$$\boldsymbol{\sigma}(t) = \mathbf{F}\{\mathbf{g}(\tau)\} = \mathbf{g}(t)\mathbf{G}\{\mathbf{C}(\tau)\}\mathbf{g}^{\dagger}(t) \tag{17}$$

where $\mathbf{C}(\tau)$ is the Cauchy strain matrix at P at time τ, referred to the configuration at time t_0, defined by

$$\mathbf{C}(\tau) = \mathbf{g}^{\dagger}(\tau)\mathbf{g}(\tau). \tag{18}$$

Replacing $\mathbf{g}(\tau)$ by $\mathbf{S}\mathbf{g}(\tau)\mathbf{S}^{\dagger}$ in eqns (17) and (18) we obtain

$$\mathbf{F}\{\mathbf{S}\mathbf{g}(\tau)\mathbf{S}^{\dagger}\} = \mathbf{S}\mathbf{g}(t)\mathbf{S}^{\dagger}\mathbf{G}\{\mathbf{S}\mathbf{C}(\tau)\mathbf{S}^{\dagger}\}\mathbf{S}\mathbf{g}^{\dagger}(t)\mathbf{S}^{\dagger}. \tag{19}$$

With eqn (15) we obtain from eqns (17) and (19)

$$\mathbf{G}\{\mathbf{C}(\tau)\} = \mathbf{S}^{\dagger}\mathbf{G}\{\mathbf{S}\mathbf{C}(\tau)\mathbf{S}^{\dagger}\}\mathbf{S}. \tag{20}$$

This relation must be satisfied for all $\mathbf{S}$ in the group of transformations $\{\mathbf{S}\}$.

3. SPATIAL DESCRIPTION OF MATERIAL SYMMETRY

The deformation of bodies B and $\bar{B}$ can also be described by the dependence of the vector positions at time τ of the generic particles P and $\bar{P}$, referred to the coordinate systems x and $\bar{x}$ respectively, on their vector positions at time t

$$\begin{aligned} \mathbf{x}(\tau) &= \mathbf{f}^*[\mathbf{x}(t), \tau] = \mathbf{f}(\mathbf{X}, \tau), \\ \bar{\mathbf{x}}(\tau) &= \mathbf{f}^*[\bar{\mathbf{x}}(t), \tau] = \mathbf{f}(\bar{\mathbf{X}}, \tau). \end{aligned} \tag{21}$$

We call this description of the deformation history the *spatial* description. If the coordinate systems x and $\bar{x}$ are equivalent, then

$$\boldsymbol{\sigma}(t) = \mathbf{F}^*\{\mathbf{g}_t(\tau)\}, \qquad \bar{\boldsymbol{\sigma}}(t) = \mathbf{F}^*\{\bar{\mathbf{g}}_t(\tau)\} \tag{22}$$

where

$$\mathbf{g}_t(\tau) = \left\|\frac{\partial x_i(\tau)}{\partial x_j(t)}\right\|, \qquad \bar{\mathbf{g}}_t(\tau) = \left\|\frac{\partial \bar{x}_i(\tau)}{\partial \bar{x}_j(t)}\right\| \tag{23}$$

and

$$\mathbf{F}^*\{\mathbf{g}_t(\tau)\} = \mathbf{F}\{\mathbf{g}(\tau)\}. \tag{24}$$

From eqn $(1)_2$ it follows that

$$\hat{\mathbf{g}}_t(\tau) = \mathbf{S}\bar{\mathbf{g}}_t(\tau)\mathbf{S}^{\dagger} \tag{25}$$

where

$$\hat{\mathbf{g}}_t(\tau) = \left\| \frac{\partial \hat{x}_i(\tau)}{\partial \hat{x}_j(t)} \right\|. \tag{26}$$

From eqn $(22)_1$ it follows that

$$\hat{\boldsymbol{\sigma}}(t) = \mathbf{F}^*\{\hat{\mathbf{g}}_t(\tau)\}. \tag{27}$$

If P and $\bar{P}$ are corresponding points in B and $\bar{B}$

$$\bar{\mathbf{g}}_t(\tau) = \mathbf{g}_t(\tau). \tag{28}$$

It follows from eqns (10), $(22)_2$, (25), (27) and (28) that

$$\mathbf{S}\mathbf{F}^*\{\mathbf{g}_t(\tau)\}\mathbf{S}^{\dagger} = \mathbf{F}^*\{\mathbf{S}\mathbf{g}_t(\tau)\mathbf{S}^{\dagger}\}. \tag{29}$$

This relation must, of course, be satisfied for all transformations of group $\{\mathbf{S}\}$ describing the material symmetry. It provides the spatial form of the material symmetry condition corresponding to the material form, eqn (15). If group $\{\mathbf{S}\}$ is the full or proper orthogonal group then the material is isotropic.

We now introduce the restriction imposed on $\mathbf{F}^*$ by the assumption that if an arbitrary time-dependent rigid rotation, which leaves the configuration at time t_0 unchanged, is superposed on the assumed deformation, then the Cauchy stress at time t is changed by the amount of this rotation at time t. We have seen in Section 2 that this implies that $\mathbf{F}\{\mathbf{g}(\tau)\}$ must be expressible in the form of eqn (17). This, in turn, implies a restriction on the form of $\mathbf{F}^*\{\mathbf{g}_t(\tau)\}$ which can be easily obtained.

From eqn (24) we find, by using the relation

$$\mathbf{g}(\tau) = \mathbf{g}_t(\tau)\mathbf{g}_t^{-1}(t_0) \tag{30}$$

that

$$\mathbf{F}^*\{\mathbf{g}_t(\tau)\} = \mathbf{F}\{\mathbf{g}_t(\tau)\mathbf{g}_t^{-1}(t_0)\}. \tag{31}$$

From eqns (18) and (30), we obtain

$$\mathbf{C}(\tau) = [\mathbf{g}_t^{-1}(t_0)]^{\dagger}\mathbf{C}_t(\tau)\mathbf{g}_t^{-1}(t_0) = \mathbf{B}_t(\tau), \text{ say} \tag{32}$$

where

$$\mathbf{C}_t(\tau) = \mathbf{g}_t^{\dagger}(\tau)\mathbf{g}_t(\tau). \tag{33}$$

Using eqns (30)–(32), we obtain from eqns (17) and (24)

$$\boldsymbol{\sigma}(t) = \mathbf{F}^*\{\mathbf{g}_t(\tau)\} = \mathbf{g}_t^{-1}(t_0)\mathbf{G}\{\mathbf{B}_t(\tau)\}[\mathbf{g}_t^{-1}(t_0)]^{\dagger}. \tag{34}$$

The restriction imposed on functional $\mathbf{G}$ by the material symmetry restriction, eqn (29), is readily obtained as

$$\mathbf{S}\mathbf{G}\{\mathbf{B}_t(\tau)\}\mathbf{S}^{\dagger} = \mathbf{G}\{\mathbf{S}\mathbf{B}_t(\tau)\mathbf{S}^{\dagger}\} \tag{35}$$

for transformations of group $\{S\}$ describing the symmetry of the material. From eqn (32) we see that this is the same restriction as eqn (15).

4. ISOTROPIC MATERIALS

If the material considered is isotropic the group of transformations $\{S\}$ is the full or proper orthogonal group. We define the matrices $\mathbf{\Pi}_\alpha$ $(\alpha = 0, 1, 2, \ldots)$ by

$$\mathbf{\Pi}_\alpha = \mathbf{B}_t(\tau_1)\ldots\mathbf{B}_t(\tau_\alpha), \qquad \mathbf{\Pi}_0 = \mathbf{I}. \tag{36}$$

It has been shown by Wineman and Pipkin[3] that for an isotropic material the functional $\mathbf{G}$ must be expressible in the form

$$\mathbf{G} = \sum_{\alpha=0}^{5} \mathbf{L}_\alpha\{\mathbf{\Pi}_\alpha\} \qquad \tau_\alpha \in [t_0, t], \tag{37}$$

where $\mathbf{L}_\alpha$ are isotropic linear operators which are themselves functionals of $\operatorname{tr}\mathbf{\Pi}_\alpha$ $(\alpha = 1, \ldots, 6)$ with $\tau_\alpha \in [t_0, t]$.

Using eqn (32) and the relation

$$\mathbf{g}_t^{-1}(t_0)[\mathbf{g}_t^{-1}(t_0)]^\dagger = \mathbf{C}_t^{-1}(t_0) \tag{38}$$

we obtain

$$\begin{aligned} \mathbf{\Pi}_\alpha &= [\mathbf{g}_t^{-1}(t_0)]^\dagger \mathbf{\Pi}_\alpha^* \mathbf{g}_t^{-1}(t_0) \\ \operatorname{tr}\mathbf{\Pi}_\alpha &= \operatorname{tr}\mathbf{\Pi}_\alpha^* \mathbf{C}_t^{-1}(t_0) \end{aligned} \tag{39}$$

where

$$\mathbf{\Pi}_\alpha^* = \mathbf{C}_t(\tau_1)\mathbf{C}_t^{-1}(t_0)\mathbf{C}_t(\tau_2)\ldots\mathbf{C}_t^{-1}(t_0)\mathbf{C}_t(\tau_\alpha). \tag{40}$$

Introducing these results into eqn (34), using eqns (37) and (38), and bearing in mind that, from the definition of a functional, $\mathbf{C}_t^{-1}(\tau_0)$ is a functional of $\mathbf{C}_t(\tau)$, we find that $\boldsymbol{\sigma}(t)$ must be expressible as an isotropic functional of $\mathbf{C}_t(\tau)$

$$\boldsymbol{\sigma}(t) = \mathbf{G}^*\{\mathbf{C}_t(\tau)\} \tag{41}$$

where $\mathbf{G}^*$ satisfies the relation

$$\mathbf{S}\mathbf{G}^*\{\mathbf{C}_t(\tau)\}\mathbf{S}^\dagger = \mathbf{G}^*\{\mathbf{S}\mathbf{C}_t(\tau)\mathbf{S}^\dagger\} \tag{42}$$

for all orthogonal $\mathbf{S}$.

It is evident that the mere fact that we choose to use a spatial description of the deformation, rather than a material description, in the constitutive equation for the Cauchy stress in an isotropic material, places no restriction on the rheological character of the material considered. We will see in Section 6 that this disagrees with the conclusions of Noll[2].

5. ISOTROPIC MATERIALS—AN ALTERNATIVE APPROACH

We have seen that in our constitutive assumption we may employ either a material or a spatial description of the deformation gradient history

$$\boldsymbol{\sigma}(t) = \mathbf{F}\{\mathbf{g}(\tau)\} = \mathbf{F}^*\{\mathbf{g}_t(\tau)\}. \tag{43}$$

If when an arbitrary time-dependent rigid rotation, which leaves the configuration at time t_0

unchanged, is superposed on the assumed deformation, the Cauchy stress is rotated by the amount of this rotation at time t, then the functional $\mathbf{F}^*$ in eqn (43) must satisfy the relation

$$\mathbf{R}(t)\mathbf{F}^*\{\mathbf{g}_t(\tau)\}\mathbf{R}^\dagger(t) = \mathbf{F}^*\{\mathbf{R}(\tau)\mathbf{g}_t(\tau)\mathbf{R}^\dagger(t)\} \tag{44}$$

for all proper orthogonal matrices $\mathbf{R}(\tau)$ such that $\mathbf{R}(t_0) = \mathbf{I}$. If the material is isotropic then (cf. eqn (29))

$$\mathbf{S}\mathbf{F}^*\{\mathbf{g}_t(\tau)\}\mathbf{S}^\dagger = \mathbf{F}^*\{\mathbf{S}\mathbf{g}_t(\tau)\mathbf{S}^\dagger\} \tag{45}$$

for all (time-independent) orthogonal matrices S. Replacing $\mathbf{g}_t(\tau)$ in eqn (45) by $\mathbf{R}(\tau)\mathbf{g}_t(\tau)\mathbf{R}^\dagger(t)$, we obtain

$$\mathbf{S}\mathbf{F}^*\{\mathbf{R}(\tau)\mathbf{g}_t(\tau)\mathbf{R}^\dagger(t)\}\mathbf{S}^\dagger = \mathbf{F}^*\{\mathbf{S}\mathbf{R}(\tau)\mathbf{g}_t(\tau)\mathbf{R}^\dagger(t)\mathbf{S}^\dagger\}. \tag{46}$$

With eqn (44) this yields

$$\mathbf{T}(t)\mathbf{F}^*\{\mathbf{g}_t(\tau)\}\mathbf{T}^\dagger(t) = \mathbf{F}^*\{\mathbf{T}(\tau)\mathbf{g}_t(\tau)\mathbf{T}^\dagger(t)\} \tag{47}$$

where

$$\mathbf{T}(\tau) = \mathbf{S}\mathbf{R}(\tau). \tag{48}$$

We note that $\mathbf{T}(\tau)$ is an arbitrary time-dependent orthogonal matrix, which is not necessarily the unit matrix for either $\tau = t_0$ or t.

If $\mathbf{T}(t) = \mathbf{I}$, then condition (47) yields

$$\mathbf{F}^*\{\mathbf{g}_t(\tau)\} = \mathbf{F}^*\{\mathbf{T}(\tau)\mathbf{g}_t(\tau)\}. \tag{49}$$

It can easily be shown that this implies that $\boldsymbol{\sigma}(t)$ must be expressible in the form

$$\boldsymbol{\sigma}(t) = \mathbf{F}^*\{\mathbf{g}_t(\tau)\} = \mathbf{G}^*\{\mathbf{C}_t(\tau)\} \tag{50}$$

where

$$\mathbf{C}_t(\tau) = \mathbf{g}_t^\dagger(\tau)\mathbf{g}_t(\tau). \tag{51}$$

Conversely, if $\mathbf{F}^*$ is expressible in the form of eqn (50) condition (49) is satisfied. Further, from eqns (45), (50) and (51) it follows that $\mathbf{G}^*\{\mathbf{C}_t(\tau)\}$ must satisfy the relation

$$\mathbf{S}\mathbf{G}^*\{\mathbf{C}_t(\tau)\}\mathbf{S}^\dagger = \mathbf{G}^*\{\mathbf{S}\mathbf{C}_t(\tau)\mathbf{S}^\dagger\} \tag{52}$$

for all constant orthogonal S.

We conclude that if in an isotropic material superposition on the assumed deformation of an arbitrary time-dependent rigid rotation causes the stress $\boldsymbol{\sigma}(t)$ to be rotated by the amount of this rotation at time t, then $\boldsymbol{\sigma}(t)$ must be expressible in the form of eqn (50) where $\mathbf{G}^*\{\mathbf{C}_t(\tau)\}$ satisfies relation (52).

Expression (50) for $\boldsymbol{\sigma}(t)$ can also be obtained from a result given by Truesdell and Noll [4, eqn (31.10)]. They show that for an isotropic material $\boldsymbol{\sigma}(t)$ must be expressible in the form

$$\boldsymbol{\sigma}(t) = \tilde{\mathbf{G}}\{\mathbf{C}_t(\tau), \mathbf{c}(t)\} \tag{53}$$

where $\mathbf{c}(t)$ is the Finger strain at time t defined by

$$\mathbf{c}(t) = \mathbf{g}(t)\mathbf{g}^\dagger(t). \tag{54}$$

Since $\mathbf{g}(t) = \mathbf{g}_t^{-1}(t_0)$ we obtain from eqns (54) and (38)

$$\mathbf{c}(t) = \mathbf{C}_t^{-1}(t_0). \tag{55}$$

Since $\mathbf{C}_t^{-1}(t_0)$ is a functional of $\mathbf{C}_t(\tau)$ it follows from eqn (53) that $\boldsymbol{\sigma}(t)$ must be expressible in the form of eqn (50).

6. NOLL'S MATERIAL SYMMETRY CONDITION

In Ref. [2] Noll defines a *simple material* as a material for which the Cauchy stress matrix $\boldsymbol{\sigma}(t)$ at the generic particle P of body B at time t, referred to the rectangular Cartesian coordinate system x, is given by‡

$$\boldsymbol{\sigma}(t) = \mathbf{F}\{\mathbf{g}(\tau)\} \tag{56}$$

where $\mathbf{F}\{\mathbf{g}(\tau)\}$ satisfies the condition

$$\mathbf{F}\{\mathbf{g}(\tau)\} = \mathbf{F}\{\mathbf{g}(\tau)\mathbf{S}\} \tag{57}$$

for all transformations $\mathbf{S}$ in a group $\{\mathbf{S}\}$ which is the full unimodular group or a sub-group of it. He calls the maximal group for which relation (57) is satisfied the *isotropy group* of the material.

If the isotropy group is the full orthogonal group or a sub-group of it, the material is said to be a *simple solid*. Otherwise it is a *simple fluid*.

In the case when the isotropy group is the full orthogonal group or a sub-group of it, we can compare the definition of material symmetry given by eqn (57) with relation (15). It is evident that they are not equivalent. However, it is easy to show that they become equivalent if material frame indifference is assumed. Indeed by taking eqn (57) as his starting point and introducing material frame indifference, Noll[2] obtains relation (20), which, as we have seen in Section 2, also results from eqn (15) by using material frame indifference.

The physical origin of the discrepancy between eqns (15) and (57) can be easily seen if we interpret eqn (57) in terms of bodies B and $\bar{B}$ introduced in Section 2. In order to arrive at eqn (57) we can suppose body $\bar{B}$ to be first rotated and translated so that the generic particle $\bar{P}$ which initially has vector position $\bar{\mathbf{X}}$, referred to the system $\bar{x}$, becomes coincident with the corresponding particle P of B, i.e. so that it occupies vector position $\mathbf{X}$ referred to coordinate system x. The body is then subjected to a deformation in which $\bar{P}$ occupies, at time τ, the vector position $\mathbf{x}(\tau)$ referred to the system x. The Cauchy stress matrices at the corresponding particles P and $\bar{P}$ in bodies B and $\bar{B}$, referred to the system x, then have equal values.

In the case when the isotropy group is the full unimodular group, i.e. when according to Noll's definition the material is a simple fluid, it is concluded, with material frame indifference, that $\boldsymbol{\sigma}(t)$ must be expressible in the form

$$\boldsymbol{\sigma}(t) = \mathbf{G}^*\{\mathbf{C}_t(\tau); \rho\} \tag{58}$$

where ρ is the density of the fluid at time t. It is also maintained that any constitutive equation of the form of eqn (58) satisfies condition (57) for all unimodular $\mathbf{S}$, i.e. the defining condition for a simple fluid. However, Noll then argues paradoxically that since the full orthogonal group is a sub-group of the unimodular group, a simple fluid must be isotropic and accordingly the functional $\mathbf{G}^*$ in eqn (58) must satisfy the condition (cf. eqn (52))

$$\mathbf{S}\mathbf{G}^*\{\mathbf{C}_t(\tau); \rho\}\mathbf{S}^\dagger = \mathbf{G}^*\{\mathbf{S}\mathbf{C}_t(\tau)\mathbf{S}^\dagger; \rho\} \tag{59}$$

for all orthogonal $\mathbf{S}$.

‡Strictly, in Noll's definition $\boldsymbol{\sigma}(t)$ is assumed to be a hereditary functional of $\mathbf{g}(\tau)$. If one wishes to adhere to this definition, one has merely to suppose that in eqn (56) the deformation gradient matrix is a function of the lapsed time $t-\tau$ rather than of τ. The conclusions reached in this section remain unchanged.

This conclusion disagrees with that in Sections 4 and 5 where it is seen that $\sigma(t)$ for *any* isotropic simple material may be expressed in the form of eqn (41), where $\mathbf{G}^*$ satisfies relation (42). (We note that relations (41) and (42) are the same as eqns (58) and (59), respectively, if the argument ρ is suppressed in the latter. They become identical in the case when the material is incompressible.)

In previous papers[5–9] attention has been drawn to the errors in Noll's argument which led to the conclusions stated above (see, also, the attempted refutation in Refs [10–12]). A fuller discussion of these will be given elsewhere. Here we merely remark on an internal inconsistency. If any functional $\mathbf{G}^*\{\mathbf{C}_t(\tau);\rho\}$ satisfies relation (57) for all unimodular $\mathbf{S}$, then it must also satisfy relation (57) for all orthogonal $\mathbf{S}$ without the imposition of further restrictions. Since, in deriving eqn (58), Noll has required that material frame indifference be satisfied, any constitutive functional of the form of eqn (58) should satisfy the condition for the material to be isotropic.

7. THERMAL CONDUCTION IN A RIGID BODY

We will now discuss thermal conduction in a rigid body. We will see that the procedure described in Section 2 for describing material symmetry and expressing the restrictions on the constitutive equation due to it can still be used. However, the procedure introduced by Noll and described in Section 6 is inapplicable.

As in Section 2 we consider two congruent bodies B and $\bar{B}$ to be cut from a homogeneous body, and x and $\bar{x}$ are two rectangular Cartesian coordinate systems similarly located with respect to B and $\bar{B}$. Let $\mathbf{Q}$ $(=\|Q_i\|)$ and $\mathbf{\Gamma}$ $(=\|\Gamma_i\|)$ be the heat flux vector and temperature gradient, referred to coordinate system x, at a generic particle P of B, located in vector position $\mathbf{X}$ $(=\|X_i\|)$ referred to the system x. Similarly, let $\bar{\mathbf{Q}}$ $(=\|\bar{Q}_i\|)$ and $\bar{\mathbf{\Gamma}}$ $(=\|\bar{\Gamma}_i\|)$ be the heat flux vector and temperature gradient, referred to the coordinate system $\bar{x}$, at a generic particle $\bar{P}$ of $\bar{B}$, located in vector position $\bar{\mathbf{X}}$ $(=\|\bar{X}_i\|)$ referred to system $\bar{x}$.

Let $\hat{\mathbf{Q}}$ $(=\|\hat{Q}_i\|)$ and $\mathbf{\Gamma}$ $(=\|\hat{\Gamma}_i\|)$ be the heat flux vector and temperature gradient at $\bar{P}$ referred to the coordinate system x and let $\hat{\mathbf{X}}$ $(=\|\hat{X}_i\|)$ be the vector position of $\bar{P}$ referred to system x. Then (cf. eqn $(1)_1$)

$$\hat{\mathbf{X}} = \mathbf{S}\bar{\mathbf{X}} + \mathbf{c} \tag{60}$$

and

$$\hat{\mathbf{Q}} = \mathbf{S}\bar{\mathbf{Q}}, \qquad \hat{\mathbf{\Gamma}} = \mathbf{S}\bar{\mathbf{\Gamma}}. \tag{61}$$

We now suppose that P and $\bar{P}$ are corresponding particles of B and $\bar{B}$, i.e. $\bar{\mathbf{X}} = \mathbf{X}$. Then, from eqn $(61)_2$

$$\hat{\mathbf{\Gamma}} = \mathbf{S}\mathbf{\Gamma}. \tag{62}$$

We make the constitutive assumption for body B that

$$\mathbf{Q} = \boldsymbol{\phi}(\mathbf{\Gamma}). \tag{63}$$

In general $\bar{\mathbf{Q}}$ will be a different function of $\bar{\mathbf{\Gamma}}$. However, if and only if

$$\bar{\mathbf{Q}} = \boldsymbol{\phi}(\bar{\mathbf{\Gamma}}) \tag{64}$$

coordinate systems x and $\bar{x}$ are said to be equivalent.

As in Section 2 we describe the symmetry of the material by the group of transformations $\{\mathbf{S}\}$ relating all pairs of the mutually equivalent coordinate systems which are equivalent to x.

From eqn (63) we have, for body $\bar{B}$

$$\hat{\mathbf{Q}} = \boldsymbol{\phi}(\hat{\mathbf{\Gamma}}). \tag{65}$$

We now suppose that $\bar{\Gamma} = \Gamma$ and obtain from eqns (61), (62), (64) and (65)

$$\mathbf{S}\phi(\Gamma) = \phi(\mathbf{S}\Gamma). \tag{66}$$

This relation must, of course, be satisfied for all transformations $\mathbf{S}$ in group $\{\mathbf{S}\}$.

However, if we attempt to use the procedure described in Section 6 we obtain for body $\bar{B}$ the same relation (63) as was assumed for body B; i.e. we find no restrictions on ϕ. In order to obtain appropriate symmetry restrictions we would have to introduce deformation gradients, which are totally irrelevant to the problem, as independent variables in the constitutive equation for $\mathbf{Q}$, thus allowing the use of material frame indifference.

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